PTR Walkthrough

$\dot{x}(t) = f(x(t), u(t))$

Need to make free final time so introduce

$\tau \in [0,1]$

$$\frac{dx}{dt} = \frac{d\tau}{dt}\frac{dx}{d\tau}$$

$$\frac{d}{d\tau}x(\tau) = \sigma f(x(\tau), u(\tau)) \qquad \sigma \triangleq \left(\frac{d\tau}{dt}\right)^{-1} \Rightarrow t = \sigma\tau$$

Need to linearize about reference $(\hat{x}, \hat{u}, \hat{\sigma})$

$$x'(\tau) \approx \hat{\sigma} f(\hat{x}(\tau), \hat{u}(\tau)) + \hat{\sigma} \frac{\partial f}{\partial x}\Big|_{\hat{x},\hat{u}}(x(\tau) - \hat{x}) + \hat{\sigma}\frac{\partial f}{\partial u}\Big|_{\hat{x},\hat{u}}(u(\tau) - \hat{u})$$
$$+ f(\hat{x}, \hat{u})(\sigma - \hat{\sigma})$$

$$x'(\tau) \approx A(\tau)x(\tau) + B(\tau)u(\tau) + \Sigma(\tau)\sigma + z(\tau)$$

$$A(\tau) \triangleq \hat{\sigma}\frac{\partial f}{\partial x}\Big|_{\hat{x},\hat{u}}$$

$$B(\tau) \triangleq \hat{\sigma}\frac{\partial f}{\partial u}\Big|_{\hat{x},\hat{u}}$$

$$\Sigma(\tau) \triangleq f(\hat{x}(\tau), \hat{u}(\tau))$$

$$z(\tau) \triangleq -A(\tau)\hat{x}(\tau) - B(\tau)\hat{u}(\tau)$$

Now we need an exact discritization. We will split the time interval $\tau \in [0,1]$ into K even temporal nodes. We will define

$$d\tau = \frac{1}{K-1} \qquad \tau_k = k\,d\tau \quad \forall k \in [0, \cdots, K-1]$$

$$\lambda^-(\tau) = \frac{\tau_{k+1} - \tau}{d\tau}$$

$$\lambda^+(\tau) = \frac{\tau - \tau_k}{d\tau}$$

Apply FOH to $u(\tau)$: $u(\tau) = \lambda^-(\tau)u_k + \lambda^+(\tau)u_{k+1}$

$$x'(\tau) = A(\tau)x(\tau) + B(\tau)\lambda^-(\tau)u_k + B(\tau)\lambda^+(\tau)u_{k+1} + \Sigma(\tau)\sigma + z(\tau)$$

The above is valid $\forall \tau \in [\tau_k, \tau_{k+1}]$

Using STM and Convolution we can write

$$x(\tau_{k+1}) = \Phi(\tau_{k+1}, \tau_k)x(\tau_k) + \int_{\tau_k}^{\tau_{k+1}} \Phi(\tau_{k+1}, \theta)\left[B(\theta)\lambda^-(\theta)u_k + B(\theta)\lambda^+(\theta)u_{k+1} + \Sigma(\theta)\sigma + z(\theta)\right] d\theta$$

We can write this as

$$x_{k+1} = A_k x_k + B_k^- u_k + B_k^+ u_{k+1} + \Sigma_k \sigma + z_k \qquad \forall k \in [0, \cdots, K-2]$$

Where

$$A_k = \Phi(\tau_{k+1}, \tau_k)$$

$$B_k^- = A_k \int_{\tau_k}^{\tau_{k+1}} \Phi^{-1}(\theta, \tau_k) B(\theta) \lambda^-(\theta) d\theta$$

$$B_k^+ = A_k \int_{\tau_k}^{\tau_{k+1}} \Phi^{-1}(\theta, \tau_k) B(\theta) \lambda^+(\theta) d\theta$$

$$\Sigma_k = A_k \int_{\tau_k}^{\tau_{k+1}} \Phi^{-1}(\theta, \tau_k) \Sigma(\theta) d\theta$$

$$z_k = A_k \int_{\tau_k}^{\tau_{k+1}} \Phi^{-1}(\theta, \tau_k)^{-1} z(\theta) d\theta$$

To compute the integrals we must use RK4 integration over each interval $[\tau_k, \tau_{k+1}]$ using N_{sub} integration steps

$$\dot{P}(\tau) = \begin{bmatrix} f(P_x(\tau), \hat{u}(\tau), \hat{\sigma}) \\ A(\tau) P_\Phi(\tau) \\ P_\Phi(\tau)^{-1} \lambda_k^-(\tau) B(\tau) \\ P_\Phi(\tau)^{-1} \lambda_k^+(\tau) B(\tau) \\ P_\Phi(\tau)^{-1} \Sigma(\tau) \\ P_\Phi(\tau)^{-1} z(\tau) \end{bmatrix} \quad P(\tau_k) = \begin{bmatrix} \hat{x}_k \\ \mathrm{flat}(I_{n_x}) \\ 0_{n_x n_u \times 1} \\ 0_{n_x n_u \times 1} \\ 0_{n_x \times 1} \\ 0_{n_x \times 1} \end{bmatrix} \quad P(\tau) = \begin{bmatrix} P_x(\tau) \\ P_\Phi(\tau) \\ P_{B^-}(\tau) \\ P_{B^+}(\tau) \\ P_\Sigma(\tau) \\ P_z(\tau) \end{bmatrix}$$

$$\dot{\Phi}(\tau, \tau_k) = A(\tau) \Phi(\tau, \tau_k)$$

$$A_k = \Phi(\tau_{k+1}, \tau_k) = \int_{\tau_k}^{\tau_{k+1}} A(\tau) \Phi(\tau, \tau_k) d\tau \qquad \Phi(\tau_k, \tau_k) = I$$

$$P_\Phi(\tau) \triangleq \Phi(\tau, \tau_k)$$

- The rest of the initial conditions are zeros because $\bar{B}_{K}, \ldots$ are all zeros for the transition from τ_{K} to τ_{K}
- The reference state is integrated with the reference control to measure dynamic feasibility.
- The integration must be performed for all intervals $[\tau_{K}, \tau_{K+1}]$ this might be parallelizable
- For a given interval $[\tau_{K}, \tau_{K+1}]$ and evaluation of the derivative $\dot{P}(\tau)$ the LU factorization can be computed for $P_{\phi}(\tau)$ and used to evaluate $P_{\phi}(\tau)^{-1}$
- All matrices are stored in vector form

Now we introduce trust regions and dynamic relaxations

Trust regions are in place to prevent unboundedness by enforcing that the new iterate of (x, u, σ) stays close to its previous iterate.

$$\delta x_K^i \triangleq x_K^i - x_K^{i-1}$$

$$\delta u_K^i \triangleq u_K^i - u_K^{i-1}$$

$$\delta \sigma^i \triangleq \sigma^i - \sigma^{i-1}$$

We will now add the following constraints with $\Delta^i \in \mathbb{R}^K$ and $\Delta_\sigma^i \in \mathbb{R}$

$$\|\delta x_K^i\|_2^2 + \|\delta u_K^i\|_2^2 \leq \Delta_K^i$$

$$\|\delta \sigma^i\|_2^2 \leq \Delta_\sigma^i$$

We then append $W_\Delta^i \|\Delta^i\| + W_{\Delta_\sigma} \|\Delta_\sigma^i\|$ to the cost function which

sigma

makes the trust region soft

Dynamic relaxation is to ensure the subproblem is dynamically feasible by introducing virtual control ν_k. The dynamics now becomes

$$x^i_{k+1} = A^i_k x^i_k + B^{-,i}_k u^i_k + B^{+,i}_k u^i_{k+1} + \Sigma^i_k \sigma^i + \nu^i_k$$

We can define $\nu^i \in \mathbb{R}^{K-1}$ as follows

$$\nu^i = [\nu^{iT}_0 \cdots \nu^{iT}_{K-2}]^T$$

We then append $W_\nu \|\nu^i\|_1$ to the cost function to drive the virtual control to zero which makes the final solution dynamically feasible