Solid Motor Combustion Model

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C_f= thrust coefficient

τ= thrust

$\dot{m}$= mass flow rate

p_0= chamber pressure

ρ_0= density of gas in combustion chamber

A^*= nozzle throat area

V_0= free volume in combustion chamber

T_0= temperature in combustion chamber

R= specific gas constant of gas

γ= ratio of specific heat of gas
N= number of grains

D= outer diameter of grain

d= inner diameter of grain

d_0= initial inner diameter of grain

L= length of grain

L_0= length of grain

A_b= grain burn area

ρ_p= density of propellant grain

a= empirical constant from St. Robert's Law

n= empirical constant from St. Robert's Law
r= grain burn rate (surface regression rate)

This document outlines the derivation for the system of differential equations that need to be solved to get chamber pressure in a solid motor given grain geometry, propellant properties, and nozzle throat area. From chamber pressure, thrust and impulse can be easily calculated.

First we will start with a mass balance.

$$\dot{m_{gen}} = \dot{m_{exit}} + \frac{dM}{dt} \tag{1}$$

Now we will analyze $\frac{dM}{dt}$, which is the rate at which mass of combustion product is accumulated in the combustion chamber.

$$\frac{dM}{dt} = \frac{d}{dt}(\rho_0 V_0) = V_0 \frac{d\rho_0}{dt} + \rho_0 \frac{dV_0}{dt} \tag{2}$$

We can then substitute in the rate of change of free volume

$$\frac{dM}{dt} = \frac{d}{dt}(\rho_0 V_0) = V_0 \frac{d\rho_0}{dt} + \rho_0 A_b r \tag{3}$$

We will assume that choked flow has been achieved, so we can plug the choked mass flow rate for $\dot{m}_{exit}$ in equation 1.

We can also plug in an expression for $\dot{m}_{gen}$ in equation 1.

Now equation 1 becomes

$$\rho_p A_b r = p_0 A^* \sqrt{\frac{\gamma}{RT_0}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} + V_0 \frac{d\rho_0}{dt} + \rho_0 A_b r \tag{4}$$

The ideal gas law is

$$p_0 = \rho_0 RT_0 \tag{5}$$

Since the chamber temperature is essentially constant during the burn, we can differentiate the expression and rearrange to get

$$\frac{V_0}{RT_0}\frac{dp_0}{dt} = r A_b(\rho_p - \rho_0) - p_0 A^* \sqrt{\frac{\gamma}{RT_0}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} \tag{6}$$

Since the propellant density is much greater than the combustion gas density, we can rewrite equation 6 as

$$\frac{V_0}{RT_0}\frac{dp_0}{dt} = r A_b \rho_p - p_0 A^* \sqrt{\frac{\gamma}{RT_0}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} \tag{7}$$

Equation 7 is the differential equation that needs to be solved to get chamber pressure in a solid motor. However, a lot of the terms in this equation are not constant. For example, the free volume of the combustion chamber increases with time as the grains burn away. Additionally, the burn area changes over time as the grains burn as well.

The derivative of the free volume with respect to time can be easily seen to be

$$\dot{V}_0 = A_b r \tag{8}$$

Below is the picture of a BATES grain from nakka-rocketry.net. The burning area are the two ends, and the inner surface.
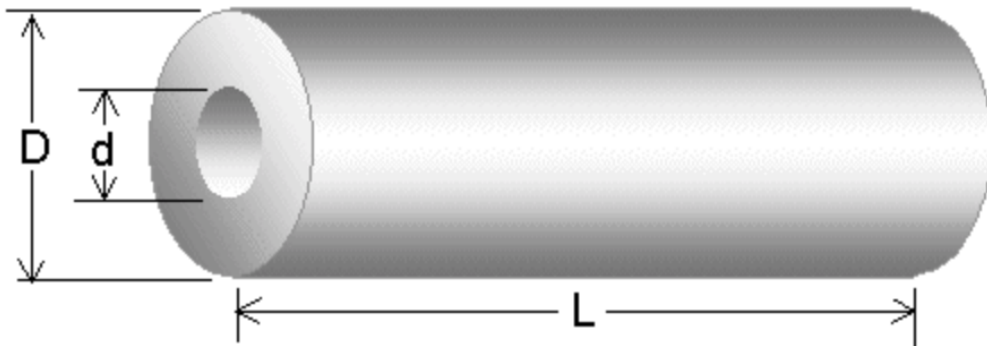
As the grain burns, we can see that the length of the grain shrink and the inner diameter increases. These rates can are given by the two equations below.

$$\dot{L} = -2r \tag{9}$$

$$\dot{d} = 2r \tag{10}$$

The reason for the 2, is because the grain burns from both ends, and the inner diameter grows radially in both directions.

Now all that is left is to model the burn area. From geometry, the burn area can be seen to be

$$A_b = N \left[\frac{\pi}{2}(D^2 - d^2) + \pi dL \right] \tag{11}$$

Taking a time derivative and and keeping in mind that d and L are both functions of time, we get

$$\dot{A}_b = N \left[-\pi d\dot{d} + \pi L\dot{d} + \pi d\dot{L} \right] \tag{12}$$

Applying equations 9 and 10, we get

$$\dot{A}_b = 2\pi r N (L - 2d) \tag{13}$$

We can now use Matlab's ode45 function on this system of differential equations. The state vector is

$$z = \begin{bmatrix} p_0 \\ V_0 \\ A_b \\ d \\ L \end{bmatrix} \tag{14}$$

The initial state vector is

$$z_0 = \begin{bmatrix} 3p_{atm} \\ \frac{NL_0\pi}{2}(D^2 - d_0^2) \\ \frac{N\pi}{2}(D^2 - d_0^2) + \pi d_0 L_0 \\ d_0 \\ L_0 \end{bmatrix} \tag{15}$$

The initial pressure for the program can really be anything. Steady-state conditions are achieved so quickly that the initial pressure doesn't really matter, but for my program, I chose to use three time atmospheric pressure. Initial free volume and burn area are calculated using geometry, and initial core diameter and initial grain length are inputs to the program.

The derivative of the state vector is

$$\dot{z} = \begin{bmatrix} \frac{RT_0}{V_0} \left[rA_b\rho_p - p_0 A^* \sqrt{\frac{\gamma}{RT_0} \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{\gamma-1}}} \right] \\ A_b r \\ 2\pi rN(L-2d) \\ 2r \\ -2r \end{bmatrix} \tag{16}$$

In the function that is inputted into ode45, the burn rate r needs to be calculated every time step using St. Robert's Law, $r = ap_0^n$

The iterations are stopped when the core diameter equals the grain outer diameter or when the length of the grain becomes 0, whichever comes first. Both of those conditions indicate that the grain has been completely burned through.

Thrust can then be calculated for each time step using the formula

$$\tau = C_f p_0 A^* \tag{17}$$

Thrust can then be numerically integrated over the motor burn time to yield total impulse of the motor.

This model was created as a proof of concept of using ode45 to do combustion modeling. In the future, this model can be refined to work with arbitrary grain geometries instead of only working for BATES grains.

Below, I have the output for a simulation. In this simulation, I have one Ammonium Perchlorate Composite Propellant grain with an outer diameter of 4 inches, an initial core diameter of 2 inches, and an initial length of 8 inches.
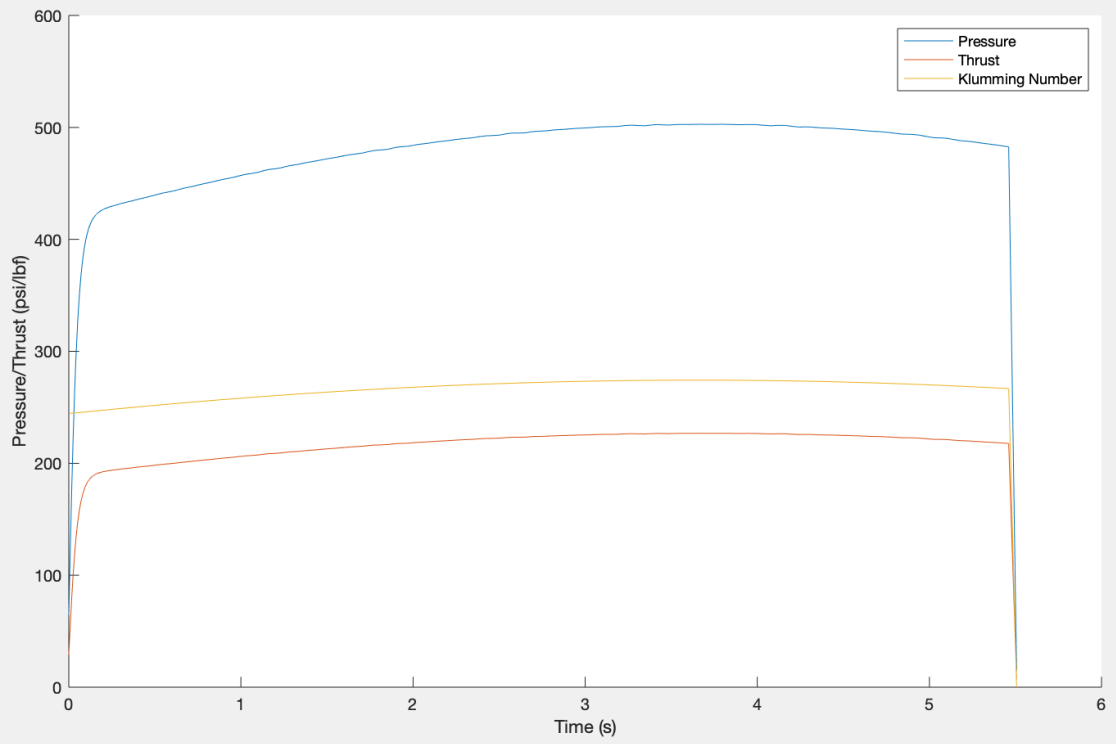
Command Window

```
Impulse= 5287 Ns
Burn Time= 5.51 secs
Max Thrust= 226.79 lbf
Max Pressure= 502.78 psi>>
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